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Maximum-Norm Interior Estimates for Ritz-Galerkin Methods

By James H. Bramble, Joachim A. Nitsche and Alfred H. Schatz

Abstract. In this paper we obtain, by simple means, interior maximum-norm estimates for a class of Ritz-Galerkin methods used for approximating solutions of second order elliptic boundary value problems in $\mathbf{R}^N$. The estimates are proved when the approximating subspaces are any of a large class of piecewise polynomial subspaces which we assume here to be defined on a uniform mesh on the interior domain. Optimal rates of convergence are obtained.

1. Introduction. In this paper we shall be concerned with obtaining "quasi-optimal" interior maximum-norm error estimates for a class of Ritz-Galerkin methods used for approximating solutions of elliptic boundary value problems. Interior estimates in L_2 Sobolev norms for such methods were obtained by Nitsche and Schatz in [15] under rather mild assumptions on the approximating subspaces. Here we shall study locally, the rate of convergence in maximum-norm using a large class of piecewise polynomial subspaces which we require to be defined on a "uniform mesh" (on the region in which the error is being estimated). We shall obtain these estimates utilizing the theory developed in [15] with the aid of a new Sobolev type inequality (proved in Section 4) which is valid for the subspaces considered here but which is in general not valid over all (sufficiently smooth functions). It will allow us to obtain the best possible rate of convergence locally for the particular subspaces used. Thomée and Westergren [18], obtained interior maximum-norm estimates involving elliptic difference operators from discrete L_2 estimates using a discrete Sobolev inequality.

An outline of the paper is as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we introduce the approximating subspaces and discuss the properties needed. In [14] and [15] a new property was introduced which was verified in [15] for some particular subspaces. The same method of verification may be used for a large class of piecewise polynomial subspaces for which a projector can be defined on each element. This fails for example in the case of smooth splines. In the appendix we shall verify this property for the tensor products of a large class of one dimensional piecewise polynomial splines. Section 4 (Theorem 2) contains our main results on interior estimates. In Section 5 we apply the estimates to specific examples.

As in [15], this paper concerns itself with approximations to solutions of second order elliptic differential equations. However, the method given there and here can be easily generalized to treat Ritz-Galerkin methods for elliptic differential equations of order $2m$.

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2. Notation and Preliminaries. Let Ω be a bounded open set in $\mathbf{R}^N$. For m a nonnegative integer, $C^m(\Omega)$ will denote the space of real valued functions having uniformly continuous derivatives up to order m on Ω with the norm,

$$|u|_{m,\Omega} = \sum_{|\alpha| \leqslant m} \sup_{x \in \Omega} |D^\alpha u(x)|. \tag{2.1}$$

C_0^∞ will denote the infinitely differentiable functions on Ω whose support is contained in Ω. The space $H^m(\Omega)$ (respectively, $\mathring{H}^m(\Omega)$) will denote the completion of $C^\infty(\Omega)$ (respectively, $C_0^\infty(\Omega)$) with respect to the norm,

$$\|u\|_{m,\Omega} = \left(\sum_{|\alpha| \leqslant m} \left(\int_\Omega |D^\alpha u|^2 \, dx \right) \right)^{1/2}. \tag{2.2}$$

Note that $H^0(\Omega) = \mathring{H}^0(\Omega) = L_2(\Omega)$.

Let $0 < h \leqslant 1$ be a parameter, and for each h, let $\Omega_j^h \subseteq \Omega$, $j = 1, \ldots, l(h)$ be disjoint open sets such that $\overline{\bigcup_{j=1}^{l(h)} \Omega_j^h} = \bar{\Omega}$. The sets Ω_j^h form a partition of Ω. For m a nonnegative integer, $C^{m,h}(\Omega)$ and $H^{m,h}(\Omega)$ will denote the space of functions whose restrictions to Ω_j^h belong to $C^m(\Omega_j^h)$ and $H^m(\Omega_j^h)$, respectively, with the corresponding norms,

$$|u|'_{m,\Omega} = \left\{ \sup_{j=1,\ldots,l(h)} |u|_{m,\Omega_j^h} \right\} \tag{2.3}$$

and

$$\|u\|'_{m,\Omega} = \left\{ \sum_{j=1}^{l(h)} \|u\|^2_{m,\Omega_j^h} \right\}^{1/2}, \tag{2.4}$$

where for simplicity we have suppressed the dependence on h in the notation.

The space $\mathring{H}^{-m}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm,

$$\|u\|_{-m,\Omega} = \sup_{v \in C_0^\infty(\Omega)} \frac{\int_\Omega uv \, dx}{\|v\|_{m,\Omega}}. \tag{2.5}$$

We shall also make use of norms involving difference quotients. For ν a multi-integer define the translation operator

$$T_h^\nu u(x) = u(x + \nu h),$$

and the forward-difference quotients,

$$\partial_{h,j} = h^{-1}(T_h^{e_j} - \mathrm{T})u.$$

Here I is the identity operator and e_j is the multi-index whose jth component is 1 and all others 0. For any multi-index α, we set

$$\partial_h^\alpha u = (\partial_{h,1}^{\alpha_1} \cdots \partial_{h,N}^{\alpha_N})u.$$

If $u \in L_2(\Omega)$ and $m \geqslant 0$ is an integer, we set

$$\|u\|_{m,\Omega,h} = \left(\sum_{|\alpha| \leq m} \int_{\Omega_{\alpha,h}} (\partial_h^\alpha u)^2 \, dx \right)^{1/2}, \tag{2.6}$$

where $\Omega_{\alpha,h}$ is the subset of Ω such that $u(x)$ is in the domain of ∂_h^α.

We shall also use Sobolev's Lemma in the following form:

LEMMA 2.1 (cf., e.g. [1]). *Let $\Omega_0 \subset\subset \Omega_1$. If $u \in H^{[N/2]+1}(\Omega_1)$, then (after possible modification of u on a set of measure zero) $u \in C(\Omega_0)$ and*

$$|u|_{0,\Omega_0} \leq C\|u\|_{[N/2]+1,\Omega_1} \tag{2.7}$$

where $C = C(\Omega_0, \Omega_1)$ and $[N/2]$ is the integral part of $N/2$.

3. The Approximating Subspaces. The local error estimates given in [15], which we will use in this paper, depend on certain properties of the approximating subspaces used. For simplicity in presentation we shall consider here piecewise polynomial subspaces defined on a uniform mesh. We proceed in the following manner: First we shall describe, in general terms, a partition of an arbitrary domain Ω, which is the restriction to Ω of a uniform partition of $\mathbf{R}^N$. Then a general class of piecewise polynomial subspaces will be defined relative to such a partition, and the properties we need will be stated. Examples will be then given.

Let Q be a given bounded simply connected domain in $\mathbf{R}^N$ and suppose that it is partitioned into disjoint open sets π_j, $j = 1, \ldots, l$, i.e. with $\overline{\bigcup_{j=1}^l \pi_j} = \bar{Q}$. For any $0 < h \leq 1$, we set $Q^h = hQ$, $\pi_j^h = h\pi_j$, and for any multi-integer ν, let $Q^{h,\nu}$ and $\pi_j^{h,\nu}$ denote the translates of Q^h and π_j^h, respectively, by $h\nu$. We shall assume that $\{Q^{1,\nu}\}$ form a uniform partition of $\mathbf{R}^N$, i.e. the $Q^{1,\nu}$ are disjoint and $\overline{\bigcup_\nu Q^{1,\nu}} = \mathbf{R}^N$. Then so do the $\{\pi_j^{h,\nu}\}$. For any bounded open set Ω, we now form the spaces $C^{m,h}(\Omega)$ and $H^{m,h}(\Omega)$ as in Section 2 using the sets $\pi_j^{h,\nu} \cap \Omega$ for the subdivision $\{\Omega_j^h\}$. T will be called a mesh domain if $\bar{T}$ is the union of some of the $\bar{\pi}_j^{h,\nu}$. The set of all mesh domains will be denoted by $\digamma_h$. We note that if $\Omega_1 \subset\subset \mathbf{R}^N$, then there exists a $\bar{T} \in \digamma_h$ such that $\bar{\Omega}_1 \subseteq \bar{T}$ and $\operatorname{dist}(\bar{T}, \bar{\Omega}_1) \leq Ch$, for h sufficiently small where C is independent of h.

In general, the finite element spaces with which we shall work will be based on a given finite set A of multi-indices and a space L spanned by the monomials $x^\beta = x_1^{\beta_1} \cdots x_N^{\beta_N}$ for $\beta \in A$; i.e.

$$L = \left\{ \varphi | \varphi = \sum_{\beta \in A} r_\beta x^\beta, \, r_\beta \in \mathbf{R} \right\}. \tag{3.1}$$

We impose the condition on A that L be translation invariant; i.e. if $\varphi(x) \in L$ and $y \in \mathbf{R}^N$, then $\varphi(x + y) \in L$. Typical examples are $A = \{\beta | \beta_i < r\}$ or $A = \{\beta \, | \, |\beta| < r\}$ for some fixed integer r. For any domain, Ω', $L_{\Omega'}$ will denote the restriction of L to Ω'. We first define a general (translation invariant) space $S^h(T)$ for any mesh domain T by

$$S^h(T) = \{\varphi | \varphi|_{\pi_j^{h,\nu}} \in L_{\pi_j^{h,\nu}} \text{ for all } \pi_j^{h,\nu} \subseteq T\}.$$

$S^h(\Omega)$ for arbitrary Ω is then the restriction of $S^h(T)$ to Ω. We note that in general the elements of $S^h(\Omega)$ may be discontinuous across the boundaries of the $\pi_j^{h,\nu}$.

The spaces we will use will be subspaces of $S^h(\Omega)$. Before defining them we shall note some properties of the general space $S^h(\Omega)$ and a property of $S^h(\Omega)$ for a particular choice of A which will be useful later on.

PROPOSITION 1. *Let $T \subset \Omega$ and $T \in \mathcal{F}_h$. Then for any $\varphi \in S_r^h(\Omega)$ inverse relations of the type,*

$$|\varphi|'_{0,T} \leqslant Ch^{-N/2} \|\varphi\|_{0,T} \tag{3.2}$$

and

$$\|\varphi\|'_{m,T} \leqslant Ch^{k-m} \|\varphi\|'_{k,T}, \qquad 0 \leqslant k \leqslant m \leqslant \max|\beta|, \beta \in A, \tag{3.3}$$

hold, where C is independent of h, φ and T.

These properties are well known and much used. The proof in the one dimensional case was given by Nitsche in [12]. The proof for the above cases are similar.

PROPOSITION 2. *Let $\Omega_0 \subset\subset \Omega_1$ and $S^h(\Omega_1)$ be as above with $\{\beta \,|\, |\beta| < r\} \subset A$ for $r \geqslant 1$ a given integer. Let α be any multi-index with $|\alpha| \leqslant t$ for t fixed but arbitrary. Then for any $u \in C^{l+|\alpha|}(\Omega_1)$, there exists a $U_h \in S^h(\Omega_1)$ such that for all h sufficiently small*

$$|\partial_h^\alpha(u - U_h)|'_{0,\Omega_0} \leqslant Ch^l |u|_{l+|\alpha|,\Omega_1}, \qquad 0 \leqslant l \leqslant r, \tag{3.4}$$

and

$$\|\partial_h^\alpha(u - U_h)\|'_{0,\Omega_0} \leqslant Ch^l \|u\|_{l+|\alpha|,\Omega_1}, \qquad 0 \leqslant l \leqslant r, \tag{3.5}$$

where C is independent of u, h and α.

The proof will be given in the appendix.

We shall now define the subspaces with which we shall work. Let $r \geqslant 2$ be an integer. For each $0 < h \leqslant 1$, $S_r^h(\Omega_1)$ will denote a subspace of $C^0(\Omega_1) \cap S^h(\Omega_1)$ having the following properties for all h sufficiently small:

A. 1. Let $\mathring{S}_r^h(\Omega_1) = \{\varphi \in S_r^h(\Omega_1) \,|\, \operatorname{supp}(\varphi) \subset \Omega_1\}$, $\Omega_0 \subset\subset \Omega_1$, and α be any fixed multi-integer. Then $\partial_h^\alpha \varphi \in \mathring{S}_r^h(\Omega_1)$ for all $\varphi \in \mathring{S}_r^h(\Omega_0)$.

A. 2. Let $\Omega_0 \subset\subset \Omega_1$. Then for each $u \in \mathring{H}^j(\Omega_0)$ there exists an $\eta \in \mathring{S}_r^h(\Omega_1)$ such that

$$\|u - \eta\|_{1,\Omega_1} \leqslant Ch^{j-1} \|u\|_{j,\Omega_0}, \qquad 1 \leqslant j \leqslant r, \tag{3.6}$$

where C is independent of u and h.

A. 3. If $\varphi \in S_r^h(\Omega_1)$, $\Omega_0 \subset\subset \Omega_1$, $\omega \in C_0^\infty(\Omega_0)$ then there exists an $\eta \in \mathring{S}_r^h(\Omega_1)$ such that

$$\|\omega\varphi - \eta\|_{1,\Omega_1} \leqslant Ch \|\varphi\|_{1,\Omega_0}, \tag{3.7}$$

where C is independent of h and φ.

Examples. In all of our examples we take $Q = \{x \,|\, 0 \leqslant x_i < 1\}$ the unit cube in $\mathbf{R}^N$.

Example 1. First let $N = 1$ and $Q = \pi_1$ (i.e. the unit interval). Then the $\{Q^{h,\nu}$ form a uniform partition of the line into intervals of length h. Let S^h be as above with $A = \{\beta = \beta_1 | \beta_1 < r\}$ and k be an integer $0 \leqslant k \leqslant r - 2$. We define $S_r^h(\mathbf{R}^1) = S^h \cap C^k$ (the piecewise polynomials of degree $\leqslant r - 2$ which belong to C^k). For $N \geqslant 2$ we define $S_r^h(\mathbf{R}^N) = S_r^h(\mathbf{R}^1) \otimes \cdots \otimes S_r^h(\mathbf{R}^1)$, the N fold tensor product of one dimensional splines and for arbitrary $\Omega \subset\subset \mathbf{R}^N$, $S_r^h(\Omega)$ is the restriction of $S_r^h(\mathbf{R}^N)$ to Ω.

Another way of describing this space directly in $\mathbf{R}^N$ for $N \geqslant 1$ is as follows: Le $Q = \pi_1$ (the unit cube in $\mathbf{R}^N$), and let S^h be taken with $A = \{\beta | \beta_i < r\}$. Then $S_r^h(\Omega)$ is the restriction to Ω of the subspace of S^h which has the property that $\varphi \in S_r^h(\Omega)$ if all the derivatives $D^\beta \varphi$ with $\beta_i \leqslant k$ are continuous on Ω. We note that when $k = r - 2$ these are the B-splines of Schoenberg [16]. For $m \geqslant 1$, an integer, $r = 2m$ and $k = m - 1$, these are piecewise Hermite splines.

Example 2. Let $N = 2$ and partition Q into triangles π_j, $j = 1, \ldots, l$, and take $S_r^h(\Omega)$ to be the elements of Bramble and Zlámal [5]. Briefly, $r = 4m + 2$, $m = 0, 1, \ldots$, etc., $A = \{\beta | \, |\beta| < r\}$ and $S_r^h(\Omega) \subset C^0(\Omega)$.

Example 3. For general N, partition Q into l n-dimensional simplices π_j and take $S_r^h(\Omega)$ to be the restriction to Ω of the continuous piecewise linear functions on this partition. Here $r = 2$.

We note that A. 1 is trivially valid and A. 2 is well known for all of the above examples.

The property A. 3 was introduced in [14]. In [15] it was verified for Example 1 (in the special case of Hermite splines) and Example 2. The proof for Example 3 follows in the same manner. The proof of A. 3 for the large class of spaces described in Example 1 will be given in the appendix.

4. Interior Estimates. Let Ω_1 be a bounded open set in $\mathbf{R}^N$ and $B(u, v)$ be a bilinear form defined on $H^1(\Omega_1) \times H^1(\Omega_1)$ of the form,

$$B(u, v) = \int_{\Omega_1} \left(\sum_{i,j=1}^N a_{ij}(x) D_i u D_j v + \sum_{i=1}^N b_i(x)(D_i u)v + c(x)uv \right) dx, \tag{4.1}$$

where for simplicity the coefficients a_{ij}, b_i and c are assumed to be of class $C^\infty(\Omega_1)$. We note that in general $B(u, v)$ may not be symmetric. We assume throughout that $B(u, v)$ is uniformly elliptic on Ω_1; i.e. there exists a constant $c > 0$ such that for all $x \in \overline{\Omega_1}$ and all real vectors $\xi = (\xi_1, \ldots, \xi_N) \neq 0$,

$$c \sum_{i=1}^N \xi_i^2 \leqslant \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j.$$

We shall be interested in obtaining estimates in the maximum norm for $u - u_h$, where $u_h \in S_r^h(\Omega_1)$ and $u \in H^1(\Omega_1)$ satisfy the "interior equations,"

$$B(u - u_h, \varphi) = 0 \quad \text{for all } \varphi \in \mathring{S}_r^h(\Omega_1). \tag{4.2}$$

Here u_h may be thought of as an approximation to u on Ω_1 obtained by using some

Ritz-Galerkin method on a larger set Ω (cf. [15]).

The following L_2 estimates were obtained in [15, Theorem 6.1].

LEMMA 4.1. *Let $\Omega_0 \subset\subset \Omega_1$ and t and p be fixed but arbitrary nonnegative integers. Let $S_r^h(\Omega)$ satisfy* A. 1, A. 2 *and* A. 3. *If $1 \leqslant j \leqslant r$, $u \in H^{t+l}(\Omega_1)$ and $u_h \in S_r^h(\Omega_1)$ satisfy* (4.2), *then there exists an $h_0 > 0$ such that for all $h \in (0, h_0]$*

$$\|u - u_h\|_{t,\Omega_0,h} \leqslant C(h^j\|u\|_{j+t,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}), \tag{4.3}$$

where C is independent of u and h but in general depends on Ω_0, Ω_1, t and p.

COROLLARY. *If $u_h \in S_r^h(\Omega_1)$ satisfies*

$$B(u_h, \varphi) = 0 \quad \textit{for all } \varphi \in \mathring{S}_r^h(\Omega_1),$$

then

$$\|u_h\|_{t,\Omega_0,h} \leqslant C\|u_h\|_{-p,\Omega_1}. \tag{4.4}$$

We shall use these estimates to obtain maximum-norm estimates for $u - u_h$. In order to do this, we shall first prove a Sobolev type inequality which is valid for the general space $S^h(\Omega_1)$ and hence for the subspace S_r^h. What is important here is that we shall replace the L_2 norms of derivatives on the right-hand side of (2.1) with L_2 norms of difference quotients.

LEMMA 4.2. *Let $\Omega_0 \subset\subset \Omega_1$. There exists an $h_0 \in (0, 1]$ such that for all $h \in (0, h_0]$ and all $\varphi \in S^h(\Omega_1)$*

$$|\varphi|'_{0,\Omega_0} \leqslant C\|\varphi\|_{[N/2]+1,\Omega_1,h}, \tag{4.5}$$

where C is independent of h and φ.

Proof. Let $\Omega_0 \subset\subset \Omega_0^+ \subset\subset \Omega_0^{++} \subset\subset \Omega_1^- \subset\subset \Omega_1$, and let x_0 be an arbitrary but fixed point such that $x_0 \in \pi_j^h$ for some j and $\pi_j^h \cap \Omega_0 \neq \varnothing$. We note that π_j^h need not lie entirely within Ω_0, but for h sufficiently small, $\pi_j^h \subset \Omega_0^+$. The set of points $x_0 + \nu h$, where ν is an arbitrary multi-integer, form a discrete mesh, say $M_{x_0}^h$, on $\mathbf{R}^N$ of size h. Let u be any mesh function defined on $M_{x_0}^h$. Then the following discrete Sobolev inequality (cf. [17]) holds:

$$\max_{y=x_0+\nu h\in\Omega_0^{++}} |\varphi(y)| \leqslant Ch^{N/2}\left(\sum_{|\beta|\leqslant[N/2]+1}\ \sum_{\zeta=x_0+\nu h\in\Omega_1^+} |\partial_h^\beta\varphi(\zeta)|^2\right)^{1/2}, \tag{4.6}$$

where C is independent of φ and h but may depend on Ω_0^{++}, Ω_1^+ and x_0. Now let x be any other point of π_j^h and consider the set of points of the form $x + \nu h \in \Omega_0^+$. This set can be mapped, by a simple translation into a subset of the set of points $x_0 + \nu h \in \Omega_0^{++}$. If now $\varphi \in C^{0,h}(\Omega_1)$, we apply the inequality (4.6) to the translated φ and then translating back we obtain

$$\max_{y=x+\nu h\in\Omega_0^+} |\varphi(y)| \leqslant Ch^{N/2}\left(\sum_{|\beta|\leqslant[N/2]+1}\ \sum_{\zeta=x+\nu h\in\Omega_1^-} |\partial_h^\beta\varphi(\zeta)|^2\right)^{1/2} \tag{4.7}$$

where $C = C(\Omega_0^+, \Omega_1^-)$ but in particular is independent of h, φ and the choice of

$x \in \pi_j^h$ or its translates in Ω_0^+. Integrating $\varphi^2(x)$ over π_j^h and using (4.7), we obtain

$$\int_{\pi_j^h} |\varphi^2(x)|\,dx \leqslant Ch^N\Bigg(\sum_{|\beta|\leqslant[N/2]+1}\Bigg(\sum_\nu \int_{\pi_j^{h,\nu}} |\partial_h^\beta \varphi(y)|^2\Bigg)\Bigg) \leqslant Ch^N \|\varphi\|^2_{[N/2]+1,\Omega_1,h} \tag{4.8}$$

where $C = C(\Omega_0, \Omega_1)$ and $\pi_j^{h,\nu}$ denotes a translation of π_j^h by νh.

If φ belongs to $S^h(\Omega_1)$, then from (3.2) we have

$$|\varphi|_{0,\pi_j^h} \leqslant Ch^{-N/2}\|\varphi\|_{0,\pi_j^h} \tag{4.9}$$

where C is independent of j, h and $u \in S^h$. Hence from (4.8) and (4.9)

$$|\varphi|_{0,\pi_j^h} \leqslant C\|\varphi\|_{[N/2]+1,\Omega_1,h}$$

holds for π_j^h or in fact for any one of its translates $\pi_j^{h\nu} \subset \Omega_0^+$. Since we need consider only a fixed finite number of such domains π_j^h $(j = 1, \ldots, l)$, the result (4.5) now follows.

Remark. An examination of the proof of Lemma 4.2 reveals that the only property of the subspaces $S^h(\Omega_1)$ that was needed was (3.2). Hence Lemma 4.2 is more generally valid for the set of functions $u \in C^{0,h}(\Omega_1)$ for which this property holds.

We are now in a position to prove a maximum-norm estimate.

THEOREM 1. *Let* $\Omega_0 \subset\subset \Omega_1 \subset\subset \mathbf{R}^N$ *and* A. 1, A. 2 *and* A. 3 *hold. Let* $0 \leqslant j \leqslant r$, *$\beta$ an arbitrary but fixed multi-index,* $u \in H^\gamma(\Omega)$ *with* $\gamma = j + |\beta| + [N/2] + 1$ *and* $u_h \in S_r^h(\Omega_1)$ *satisfy*

$$B(u - u_h, \varphi) = 0 \quad \textit{for all } \varphi \in \mathring{S}_r^h(\Omega_1).$$

Let p be an arbitrary but fixed nonnegative integer. Then there exists an $h_0 \in (0, 1]$ *such that for all* $h \in (0, h_0]$

$$|\partial_h^\beta(u - u_h)|_{0,\Omega_1} \leqslant C(h^j\|u\|_{\gamma,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}), \qquad 0 \leqslant j \leqslant r, \tag{4.10}$$

where C is independent of u and h.

Remark 3. When $|\beta| = 0$, (4.10) gives a maximum-norm estimate for the error $u - u_h$.

Proof. Let $\Omega_0 \subset\subset \Omega_0^+ \subset\subset \Omega_1^- \subset\subset \Omega_1$. The triangle inequality yields

$$|\partial_h^\beta(u - u_h)|_{0,\Omega_0} \leqslant |\partial_h^\beta(u - U_h)|'_{0,\Omega_0} + |\partial_h^\beta(U_h - u_h)|'_{0,\Omega_0}, \tag{4.11}$$

where we first choose an S^h as in Proposition 2 such that $S_r^h \subset S^h$; and then take $U_h \in S^h$ satisfying (3.4) and (3.5) with Ω_0 and Ω_1 there replaced by Ω_0^+ and Ω_1^-. Hence

$$|\partial_h^\beta(u - U_h)|_{0,\Omega_0} \leqslant Ch^j|u|_{l+|\beta|,\Omega_1^-} \leqslant Ch^j\|u\|_{\gamma,\Omega_1}. \tag{4.12}$$

Now $\partial_h^\beta(U_h - u_h) \in S^h(\Omega_0^+)$ and it follows from (4.5), (4.3), (3.5) and (4.12) that

$$
\begin{aligned}
|\partial_h^\beta(U_h - u_h)|_{0,\Omega_0} &\leqslant C\|u_h - U_h\|_{[N/2]+1+|\beta|,\Omega_0^+,h} \\
&\leqslant C(\|u - U_h\|_{[N/2]+1+|\beta|,\Omega_0^+,h} + \|u - u_h\|_{[N/2]+1+|\beta|,\Omega_0^+,h}) \qquad (4.13)\\
&\leqslant C(h^j\|u\|_{\gamma,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}).
\end{aligned}
$$

The inequality (4.10) now follows from (4.11), (4.12) and (4.13).

COROLLARY. *If $u_h \in S_r^h(\Omega_1)$ satisfies $B(u_h, \varphi) = 0$ for all $\varphi \in \overset{\circ}{S}{}_r^h(\Omega_1)$, then*

$$
|u_h|_{0,\Omega_0} \leqslant C\|u_h\|_{-p,\Omega_1}. \tag{4.14}
$$

We shall now consider estimates for the rate of convergence of difference quotients of u_h to derivatives of u. Consider finite difference operators of order k of the form,

$$
Q_h u = \sum_{\nu,|\beta|\leqslant k} C_{\nu_\beta} T_h^\nu \partial_h^\beta u, \tag{4.15}
$$

where ν is an arbitrary multi-integer and all but a finite number of the constants C_{ν_β} vanish. If j is a nonnegative integer and α a multi-index, we shall say that $Q_h = Q_h^\alpha$ of order $|\alpha|$ approximates D^α with order of accuracy j on Ω_0, if for all $u \in C^{j+|\alpha|}(\Omega_1)$

$$
|D^\alpha u - Q_h^\alpha u|_{0,\Omega_0} \leqslant Ch^j|u|_{j+|\alpha|,\Omega_1}, \tag{4.16}
$$

where $\Omega_0 \subset\subset \Omega_1$ and $C = C(\Omega_0, \Omega_1)$.

THEOREM 2. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \mathbf{R}^N$, A. 1, A. 2, and A. 3 hold. Let $1 \leqslant j \leqslant r$, α any fixed multi-index, $\gamma = j + |\alpha| + [N/2] + 1$, $u \in H^\gamma(\Omega_1)$ and $u_h \in S_r^h(\Omega_1)$ satisfy*

$$
B(u - u_h, \varphi) = 0 \quad \text{for all } \varphi \in \overset{\circ}{S}{}_r^h(\Omega_1).
$$

Let Q_h^α be any finite difference operator of order $|\alpha|$ of the form (4.15) which approximates D^α with order of accuracy j. Let p be an arbitrary but fixed nonnegative integer. Then there exists an $h_0 \in (0, 1]$ such that for all $h \in (0, h_0]$

$$
|D^\alpha u - Q_h^\alpha u_h|_{0,\Omega_0} \leqslant C(h^j\|u\|_{\gamma,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}), \qquad 1 \leqslant j \leqslant r, \tag{4.17}
$$

where C is independent of h and u, but in general depends on p, α, Ω_0, and Ω_1.

Proof. Let $\Omega_0 \subset\subset \Omega_0^+ \subset\subset \Omega_1$, then using the triangle inequality and the definition of Q_h^α we obtain for h sufficiently small that

$$
\begin{aligned}
|D^\alpha u - Q_h^\alpha u_h|_{0,\Omega_0} &\leqslant |D^\alpha u - Q_h^\alpha u|_{0,\Omega_0} + |Q_h^\alpha(u - u_h)|_{0,\Omega_0} \\
&\leqslant |D^\alpha u - Q_h^\alpha u|_{0,\Omega_0} + C \sum_{|\beta|\leqslant|\alpha|} |\partial^\beta(u - u_h)|_{0,\Omega_0^+}.
\end{aligned}
$$

The inequality (4.17) is now obvious in view of (4.16) and (4.10).

Remark. Previously we required that the partition and the subspaces defined on them have certain properties which are invariant with respect to translations of size h in the directions of the coordinate axes. It can be easily seen that the results of Theorem

1 (or Theorem 2) in the case $\alpha = 0$ remain valid for piecewise polynomial subspaces defined on partitions which may be mapped onto such partitions by a nonsingular affine mapping.

5. Examples. Here we shall exemplify the theory given in Section 4. We shall restrict ourselves to discussing Dirichlet's problem. Let Ω be a bounded domain in $\mathbf{R}^N$ with boundary $\partial\Omega$. For simplicity consider

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

For the purposes of the applications given here, we shall make one further approximability assumption on the subspace $S_r^h(\Omega)$. Namely there exists a constant C independent of u and h such that for all $u \in H^t(\Omega)$, $1 \leqslant t \leqslant r$,

$$\inf_{\chi \in S_r^h(\Omega)} \|u - \chi\|_{1,\Omega} \leqslant Ch^{t-1} \|u\|_{t,\Omega}. \tag{5.2}$$

We shall sometimes require that the elements of $S_r^h(\Omega)$ vanish on $\partial\Omega$. In this case we shall assume that (5.2) holds only for $u \in \mathring{H}^1(\Omega) \cap H^t(\Omega)$.

In what follows we shall suppose that the hypotheses of Theorem 2 are satisfied where $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$.

Example 1. *Dirichlet's Problem on a Smooth Domain.* In this example we shall assume for simplicity that $\partial\Omega \in C^\infty$. In Babuška [2] and Nitsche [10], methods were introduced for approximating solution of (5.1) in which the approximating subspaces need not satisfy the boundary condition. These methods have the same interior equations; i.e. if u_h is the approximate solution determined by any one of these methods then

$$\int_\Omega \sum_{i=1}^N \frac{\partial u_h}{\partial x_i} \frac{\partial \varphi}{\partial x_i}\, dx = \int_\Omega f\varphi\, dx = \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i}\, dx \quad \forall \varphi \in \mathring{S}^h(\Omega_1). \tag{5.3}$$

What is important here is that we may choose $S_r^h(\Omega)$ to satisfy the conditions of Theorem 3 (cf. [15, Example 2], for more details).

Now it was shown in [4] that the estimate,

$$\|e\|_{2-r,\Omega} \leqslant h^{r-1+t} \|u\|_{t,\Omega}, \tag{5.4}$$

is valid for $1 \leqslant t \leqslant r$. Using the inequalities (5.4), (4.17) and the fact that $\|e\|_{2-r,\Omega_1} \leqslant \|e\|_{2-r,\Omega}$, we obtain the error estimate,

$$|D^\alpha u - Q_h^\alpha u_h|_{0,\Omega_0} \leqslant Ch^r(\|u\|_{r+|\alpha|+[N/2]+1,\Omega_1} + \|u\|_{2,\Omega}). \tag{5.5}$$

Here α is any multi-index, Q_h^α is any finite difference operator of the form (4.16) approximating D^α with order of accuracy r. In terms of the data, (5.5) may be rewritten as

$$|D^\alpha u - Q_h^\alpha u_h|_{0,\Omega_0} \leqslant Ch^r(\|f\|_{r+|\alpha|+[N/2]-1,\Omega_1} + \|f\|_{0,\Omega}). \tag{5.6}$$

Example 2. *Dirichlet's Problem on the Square.* Let Ω be the unit square in $\mathbf{R}^2$, i.e. $\Omega = \{x | 0 \leqslant x_i \leqslant 1, i = 1, 2\}$. For our subspace $S_r^h(\Omega)$, we may choose any one of the examples given in Section 3 requiring that the elements vanish on the boundary. Let u_h be determined satisfying (5.3) for all $\varphi \in S_r^h(\Omega)$. Then (cf. [15, Example 3, Section 7]) the estimate (5.4) holds, and applying Theorem 2, it follows that both the error estimates (5.5) and (5.6) remain valid in this case.

6. Appendix.

Proof of Proposition 2. It is easily seen that we can restrict ourselves to the case where $\operatorname{supp}(u) \subset \Omega_1$. Let Pu be the L_2 projection of u onto $S^h(\mathbf{R}^N)$. This projection is locally determined; i.e. the restriction of Pu to each $\pi_j^{h,\nu}$ is equal to the L_2 projection of u on $S^h(\pi_j^{h,\nu})$. It follows from the Bramble-Hilbert Lemma [3] that

$$\|u - Pu\|_{0,\pi_j^{h,\nu}} \leqslant Ch^l \|u\|_{l,\pi_j^{h,\nu}}, \qquad 0 \leqslant l \leqslant r.$$

Now using (3.2) and the above inequality we have

$$|Pu|_{0,\pi_j^{h,\nu}} \leqslant Ch^{-N/2} \|Pu\|_{0,\pi_j^{h,\nu}} \leqslant Ch^{-N/2} \|u\|_{0,\pi_j^{h,\nu}} \leqslant C|u|_{0,\pi_j^{h,\nu}}.$$

Therefore

$$|u - Pu|_{0,\pi_j^{h,\nu}} \leqslant C|u|_{0,\pi_j^{h,\nu}},$$

and it follows from [3] that

$$|u - Pu|_{0,\pi_j^{h,\nu}} \leqslant Ch^l |u|_{l,\pi_j^{h,\nu}},$$

and the inequalities (3.4) and (3.5) follow easily in the case $\alpha = 0$. The case for general α follows from the case $\alpha = 0$ with the observation that $P(\partial^\alpha u) = \partial^\alpha(Pu)$.

Verification of A. 3 *in the Case of Tensor Products of One Dimensional Splines* (*Example* 1, *Section* 3). First consider the case $N = 1$ and let $I_0 = \Omega_0$ and $I_1 = \Omega_1$ be finite intervals with common midpoints. Then it follows from Douglas, Dupont and Wahlbin [7] that for any $u \in \mathring{H}^k(I_0) \cap H^{r,h}(I_0)$

$$\|u - Pu\|_{0,I_1} = \inf_{\varphi \in \mathring{S}_r^h(I_1)} \|u - \varphi\|_{0,I_1} \leqslant Ch^r \|u^{(r)}\|'_{0,I_0}. \tag{6.1}$$

We note that

$$\|u - Pu\|_{0,I_1} \leqslant \|u\|_{0,I_0}. \tag{6.2}$$

Consider now $N \geqslant 2$, and let Ω_1 be the N-fold product $\Omega_1 = I_1 \times \cdots \times I_1$. For $u \in L_2(\Omega_1)$, and each fixed value of the variables $x_i \in I_1$, $i \neq j$, $P_j u$ is defined by

$$\|u - P_j u\|_{0,I_1} = \inf_{\varphi \in \mathring{S}_r^h(I_1)} \|u - \varphi(x_j)\|_{0,I_1}.$$

That is, for each fixed value of $x_i \in I_1$, $i \neq j$, $P_j u$ is the one dimensional L_2 projection onto $\mathring{S}_r^h(I_1)$. Note that $P_j u$ is a function of $(x_1, \ldots, x_n)$. Now define $Pu = P_1 u \cdots P_N u$. Set $\Omega_0 = I_0 \times \cdots \times I_0$, with I_0 and I_1 as before (hence Ω_0 and Ω_1 are cubes with common centers and $\Omega_0 \subset \Omega_1$). If $u \in \mathring{H}^k(\Omega_0) \cap H^{r,h}(\Omega_0)$, it follows using (6.1) and (6.2) that

$$\|u - Pu\|_{0,\Omega_1} \leqslant \|u - P_1u\|_{0,\Omega_1} + \|P_1(u - P_2u)\|_{0,\Omega_1}$$

$$+ \cdots + \|P_1 \cdots P_{N-1}(u - P_Nu)\|_{0,\Omega_N} \leqslant Ch^r \sum_{j=1}^{N} \left\| \frac{\partial^r u}{\partial x_j^r} \right\|'_{0,\Omega_0}, \tag{6.3}$$

where we note that $\operatorname{supp}(Pu) \subseteq \Omega_1$.

It is not hard to see that

$$\lim_{h\to 0} \|u - Pu\|_{1,\Omega_1} = 0. \tag{6.4}$$

Consider the sequence $h_l = 2^{-l}h$ for given h and refine the given partition into cubes with side h_l. For given h_l let $P^l = P$ corresponding to the given h_l. Then in view of (6.4), (3.3) and (6.3) we have

$$\begin{aligned}
\|u - P^lu\|_{1,\Omega_1} &\leqslant \sum_{l=0}^{\infty} \|P^{l+1}u - P^lu\|_{1,\Omega_1} \\
&\leqslant C \sum_{l=0}^{\infty} h_{l+1}^{-1} \|P^{l+1}u - P^lu\|_{0,\Omega_1} \\
&\leqslant C \sum_{l=0}^{\infty} h_{l+1}^{-1} (\|u - P^lu\|_{0,\Omega_1} + \|u - P^{l+1}u\|_{0,\Omega_1}) \\
&\leqslant C \sum_{l=0}^{\infty} h_{l+1}^{-1} (h_l^r + h_{l+1}^r) \left(\sum_{j=1}^{N} \left\| \frac{\partial^r u}{\partial x_j^r} \right\|'_{0,\Omega_0} \right) \\
&\leqslant Ch^{r-1} \sum_{j=1}^{N} \left\| \frac{\partial^r u}{\partial x_j^r} \right\|'_{0,\Omega_0}.
\end{aligned} \tag{6.5}$$

Now consider any two domains $\Omega_0 \subset\subset \Omega_1 \subset \mathbf{R}^N$ with $\operatorname{dist}(\bar{\Omega}_0, \bar{\Omega}_1) = 4d$. Cover Ω_0 with a finite number of cubes C_i^d with side d and consider along with these the cubes C_i^{2d} having the same center as C_i^d. Let $\theta_i \in C_0^\infty(C_i^{2d})$ and $\Sigma_i \theta_i = 1$. Set $u_i = \theta_i u$, where $u \in \mathring{H}^k(\Omega_0) \cap H^{r,h}(\Omega_0)$. For each i we have

$$\|u_i - Pu_i\|_{1,C_i^d} \leqslant Ch^{r-1} \left\| \frac{\partial^r u_i}{\partial x_j^r} \right\|'_{0,C_i^{2d}}.$$

From this is easily follows that there exists an $\eta \in \mathring{S}_r^h(\Omega_1)$ such that

$$\|u - \eta\|_{1,\Omega_1} \leqslant Ch^{r-1} \left(\|u\|'_{r-1,\Omega_0} + \sum_{j=1}^{N} \left\| \frac{\partial^r u}{\partial x_j^r} \right\|'_{0,\Omega'_0} \right). \tag{6.6}$$

We are now in a position to prove (3.7) in this case. Let $\varphi \in S_r^h(\Omega_1)$ and $\omega \in C_0^\infty(\Omega_0)$ then $\omega\varphi \in \mathring{H}^k(\Omega_0) \cap H^{r,h}(\Omega_0)$. Applying (6.6) to $\omega\varphi = u$, we obtain

$$\|\omega\varphi - \eta\|_{1,\Omega_1} \leqslant Ch^{r-1} \|\varphi\|'_{r-1,\Omega_0},$$

where we have used Leibniz's rule, the property that $\omega \in C_0^\infty(\Omega_0)$ and the fact that

$$\sum_{j=1}^{N} \left\| \frac{\partial^r \varphi}{\partial x_j^r} \right\|'_{0,\Omega_0} = 0.$$

The inequality (3.7) now follows from the inverse property (3.3).

Department of Mathematics
Cornell University
Ithaca, New York 14850

Institut für Angewandte Mathematik
Universität Freiburg
Freiburg, Germany

Department of Mathematics
Cornell University
Ithaca, New York 14850

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